Derivation of the Higher Order Geometric Stiffness Matrix of a Tetrahedral Element Based on Rigid Body Rule and Incremental Force Equilibrium

應用剛體運動法則與增量力平衡在四面體元素高階幾何勁度矩陣的推導

J. T. Chang[1], C. C. Yeo
E-mail: jtchang@vnu.edu.tw

[1]*Department of Civil Engineering, Vanung University, Chung-Li Dist., Tao-Yuan City, Taiwan, Republic of China*

張健財[1], 姚程智[1]
[1]萬能科技大學 營建科技系
32061 桃園縣中壢市萬能路1號

Abstract

The higher order geometric stiffness matrix of a tetrahedral element is presented in this paper. The Derivation is based on the concept of rigid body motion. The element has four nodes. Each node has three degrees of freedom. The element can satisfy the requirements of the Rigid Body Rule and Incremental Force Equilibrium test. Because that the coefficients exiting in the higher order geometric stiffness matrix of the tetrahedral element are shown in explicit expressions, this makes engineers be easy to conduct it in numerical programs of the geometric nonlinear analysis of the solid structures.

Key words : higher order geometric stiffness matrix、rigid body motion

1. introduction

To state the motion, one-parameter function of time (t) of a material body moving in a Euclidean Space, is portrayed in Figure 1.1. A referential frame is always required [1,2] to formulate the status.

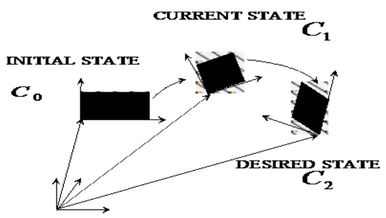

Figure 1.1 Motion of a material body

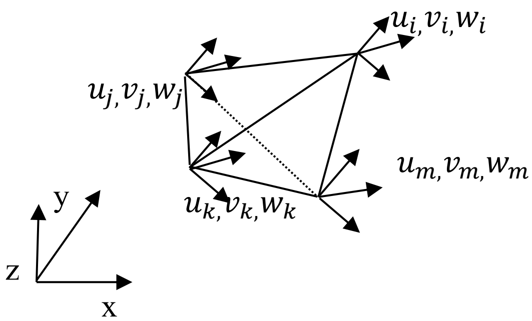

Figure 1.2 A tetrahedral element with 12 degrees of freedom

Conventionally, using initial undeformed state, or current (known) deformed state as the references are denoted total Lagrangian description, ($t = t_0$), or updated Lagrangian description at $t = t_1$ respectively. The left subscript of a symbol denotes its reference. The C_0 and C_1 denote the initial undeformed configuration and current deformed configuration respectively. By the variational principle, the approximation to the solution of the desired state is available, the product of Piola_Kirchhoff stresses and Green-Largrangian strains for more detail referred to [1]. For example, a tetrahedral element has nodal forces with three degrees of freedom per node as shown in Figure 1.2. The element nodal displacement vectors $\{q\}$ can be denoted as

$$\{q\}^T = \sum_{i=1}^{4}\{u_{i,}v_{i,}w_i\} \tag{1.1}$$

The initial force vectors $\{\,^1f\}$ is

$$\{\,^1f\}^T = \sum_{i=1}^{n-th}\{f_{xi,}f_{yi,}f_{zi,}\} \tag{1.2}$$

In general, based on the principle of virtual work. Conventionally with the aid of shape functions related to the element nodal degrees of freedom, a geometrically nonlinear tetrahedral element can be derived, it can be written as

$$\{_1^2f\} = \{_1^1f\} + [k_g]\{q\} + [\Delta k_g]\{q\} + [k_e]\{q\} \tag{1.3}$$

Matrices$[k_g]$, $[\Delta k_g]$, and$[k_e]$denote the geometrical stiffness matrix, the higher-order stiffness matrix, and the linear elastic matrix, respectively. Vectors $\{_1^2f\}$and$\{_1^1f\}$ denote the nodal force vectors of the element at C_0 and C_1, respectively.

2. Property of the geometric and higher order stiffness matrices

From Yang and Kuo [1] in 1994, they had established geometric nonlinearly elements include truss, beam and space frame elements. All of them satisfy physical intuitive requirements. Based on the rigid body rule, one can write down

$$[k_g]\{q_r\} = \{_1^2f\} - \{_1^1f\} \tag{2.1}$$

where $\{q_r\}$ represents the rigid displacements of a tetrahedral element,Let us illustrate a tetrahedral element has four nodes and its in-plane displacements which are approximated by a set of linear interpolation functions. It can be written as

$$u = L_i u_i; v = L_i v_i; w = L_i w_{i,} \tag{2.2}$$

where u_i, v_i, w_i represent the i-th nodal displacements of the tetrahedral element in the three axes. Its shape functions can be written as

$$L_i = \frac{1}{6V}(a_i + b_i x + c_i y + d_i z) \tag{2.3}$$

where

$$[V] = \frac{1}{6}det\begin{bmatrix} 1 & x_i & y_i & z_i \\ 1 & x_j & y_j & z_j \\ 1 & x_k & y_k & z_k \\ 1 & x_m & y_m & z_m \end{bmatrix}; a_i = det\begin{bmatrix} x_j & y_j & z_j \\ x_k & y_k & z_k \\ x_m & y_m & z_k \end{bmatrix}$$

$$b_i = -det\begin{bmatrix} 1 & y_j & z_j \\ 1 & y_k & z_k \\ 1 & y_m & z_k \end{bmatrix}; c_i = det\begin{bmatrix} x_j & 1 & z_j \\ x_k & 1 & z_k \\ x_m & 1 & z_k \end{bmatrix}; d_i = -det\begin{bmatrix} x_j & y_j & 1 \\ x_k & y_k & 1 \\ x_m & y_m & 1 \end{bmatrix}$$

here V represents the volume of the element. Substituting Equations (2.4) and (2.5) into (2.2), it yields

$$[k_g] = \frac{1}{6V}\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \qquad (2.4)$$

Where

$$A_{ij} =$$

$$\Big\lfloor b_j\ {}^1F_{xi} + \ c_j\ {}^1F_{yi} + d_j\ {}^1F_{zi} \qquad b_j\ {}^1F_{xi} + \ c_j\ {}^1F_{yi} + d_j\ {}^1F_{zi} \qquad b_j\ {}^1F_{xi} + \ c_j\ {}^1F_{yi} + d_j\ {}^1F_{zi} \Big\rfloor$$

(2.5)

Here we easily derive the geometric stiffness matrix of the element based on Equations (2.4) and (2.5).

In order to determine the higher order stiffness matrix, the rigid body rule should be described here first. Let us see Figure 2.1

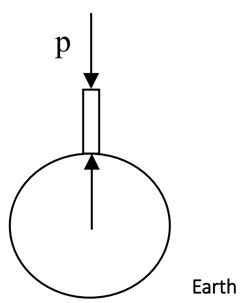

Figure 2.1(a) Before a rigid body rotation

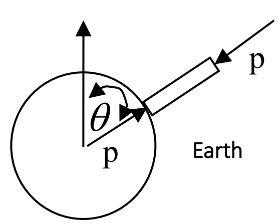

Figure 2.1(b) After a rigid body rotation

When there is a column member founded and built on surface of earth, one sees a pair of initial forces exerted on its two both end sides, i.e., Figure 2.1 (a) and after rotations, i.e., Figure 2.1 (b). One sees that the forces will accompany and follow the earth rotations without changing its magnitudes but its directions. By inspection this physical phenomena, Yang and Chiou [2] proposed the so call rigid body test to verify the nonlinear element. It can be expressed as follows

$$[k_g]\{u_r\} = \{{}_1^r f\} - \{{}_1^1 f\} \qquad (2.6)$$

$$\{{}_1^r f\} = [R]\{{}_1^1 f\} \qquad (2.7)$$

Equations (2.6) and (2.7) represent all mechanical natural behavior of materials from physical institutive concepts. $[R]$ represents a rotation matrix. With the aid of Equation (2.7), Equation (2.6) can be written as follows

$$[k_g]\{u_r\} = [R]\{{}_1^1 f\} - \{{}_1^1 f\} \qquad (2.8)$$

It can be said that initial force vectors of the element will follow the element rigid body motions without changing its magnitudes. One can set a corollary that if element has a stretch effect that will generate incremental internal forces$\{{}_1 f\}$at first and then accompany and follow rigid body rotations, one can write the following equation

$$[\Delta k_g]\{u_r\} = [R]\{{}_1 f\} - \{{}_1 f\} \qquad (2.9)$$

Equation (2.9) was used to derive the higher order geometric stiffness matrix of the framed element by Chang [3]. Herein it also holds for the motions of the tetrahedral element under stretching effects at first and accompanying by rigid body motions. Once the geometric stiffness matrix$[k_g]$ is derived, the higher order stiffness matrix of the element can be formed just merely by changing the initial forces in the geometric stiffness matrix stead of the incremental forces$\{{}_1 f\}$. One can easily establish the higher order stiffness matrix of a tetrahedral element by changing A_{ij} as ΔA_{ij} in Equation (2.4) as follows

$$\Delta A_{ij} = \begin{bmatrix} b_j\,{}_1F_{xi} + c_j\,{}_1F_{yi} + d_j\,{}_1F_{zi} & & \\ & b_j\,{}_1F_{xi} + c_j\,{}_1F_{yi} + d_j\,{}_1F_{zi} & \\ & & b_j\,{}_1F_{xi} + c_j\,{}_1F_{yi} + d_j\,{}_1F_{zi} \end{bmatrix} \qquad (2.10)$$

The derived element also satisfies the requirements of the Rigid Body Rule and Incremental Force Equilibrium test proposed by Chi and Kuo [4]. In this study the element lacks of moment effects, the equation can be written as follows.

$$[k_g]^T\{u_r\} = [R]\{{}_1^1 f\} - \{{}_1^1 f\} \qquad (2.10)$$

As for the higher order stiffness matrix of the tetrahedral element is satisfied the requirements test.

$$[\Delta k_g]^T\{u_r\} = [R]\{{}_1 f\} - \{{}_1 f\} \qquad (2.11)$$

3. Test Example

A Solid Wedge under a point load

Figure 3.1 shows a wedge under a down ward load at the apex of the wedge structure. It was modelled by 28 tetrahedral elements. The boundary conditions are as shown in Figure 3.1. Results of the analysis behaviour show improvement on the analysis of the solid wedge under a point load with a little stiff than just considerations geometrical stiffness parts in the analysis as shown in Figure 3.2.

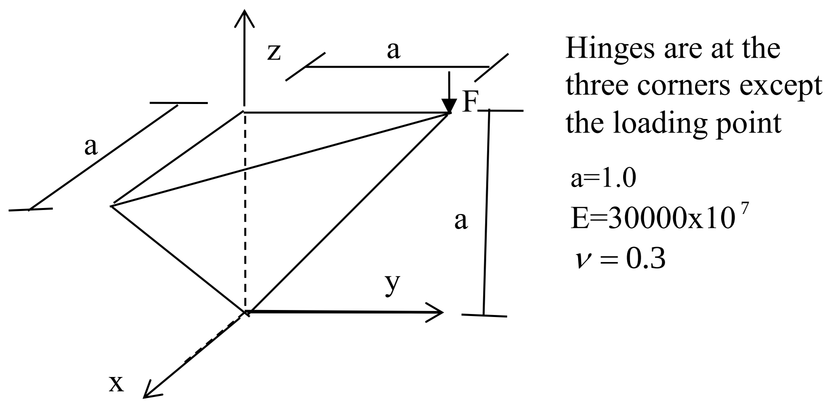

Figure 3.1 Geometry, material properties and the layout of solid wedge under a downward point load applied at the apex of the wedge

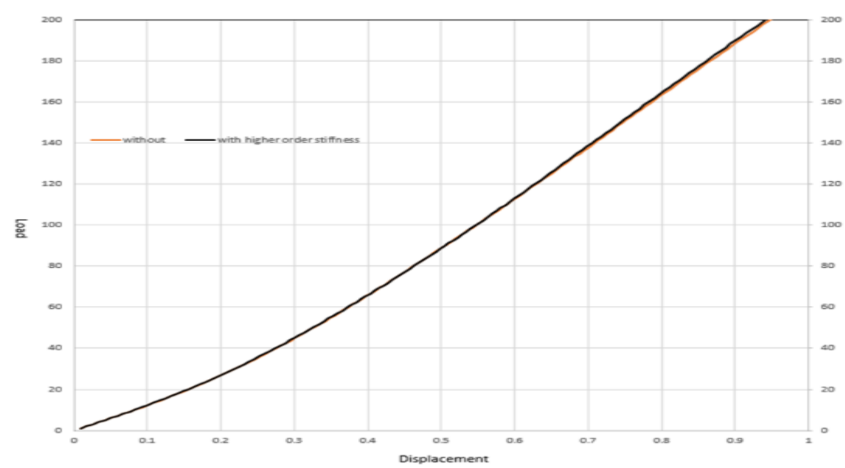

Figure 3.2 Post-buckling response of the solid wedge under a down ward point load applied at the apex of the wedge

4. Conclusion

In this paper, the explicit higher order geometric stiffness matrix of tetrahedral element can be derived based on using the concept of rigid body rule. All terms in the higher order geometric stiffness matrix of the tetrahedral element are explicit. One uses exiting geometrical stiffness matrix to derive the higher order geometrical stiffness matrix of the tetrahedral element based on physically intuitive concepts of rigid body motions. For example it shows with the higher order stiffness matrix in the analysis the structures behave little stiff than convectional using geometrical stiffness only.

Reference

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